

# Critical Behaviour of a Homogeneous Bose Gas at Finite Temperature

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A homogeneous non-relativistic Bose gas is investigated at finite temperature using renormalization group methods. The phase transition is shown to be second order and the effective chemical potential and the effective quartic coupling vanish at the critical temperature. We obtain the critical exponent  $\nu = 0.73$  at leading order in the derivative expansion.

The remarkable achievement of Bose-Einstein condensation (BEC) of alkali atoms in magnetic traps [1] has created an enormous interest in the properties of dilute Bose gases. A very recent review on trapped Bose gases can be found in [2]. The homogeneous Bose gas at zero temperature was extensively studied in the fifties [3,4]. The properties of this system can be calculated in the loop expansion which is an expansion in powers of  $\sqrt{\rho a^3}$ , where  $\rho$  is the density and  $a$  is the S-wave scattering length. The one-loop correction to the ground state energy was calculated in 1957 by Lee and Yang [3], and the two-loop contribution was recently obtained by Braaten and Nieto [5]. In Ref. [6] Haugset *et al.* have studied this system at finite temperature, and included the daisy and superdaisy diagrams by self-consistently solving a gap equation for the effective chemical potential. The inclusion of these diagrams are essential in order to satisfy the Goldstone theorem at finite temperature. Biljsma and Stoof [7] have applied the renormalization group [9] to study the homogeneous Bose gas at finite temperature. Their results clearly demonstrate that the phase transition is second order as expected, since this system is in the same universality class as three-dimensional  $xy$ -model which is known to have a second order phase transition. Moreover, the critical temperature increases by approximately 10% compared to the noninteracting Bose gas [7]. A review summarizing our current understanding of homogeneous Bose gases can be found in [8].

In the present work we reconsider the non-relativistic homogeneous spin zero Bose gas at finite temperature. We focus in particular on the critical behaviour and the calculation of critical exponents using RG techniques. The work of Biljsma and Stoof [7] only included the effects of two and three-particle interactions by neglecting the running of higher order vertices. In our work we include the RG flow of higher order particle interactions and show that the critical exponents do not converge to the expected  $O(2)$ -model results as successive terms are included.

The Bose gas can be described by an effective quantum field theory [10]. As long as the momenta  $p$  of the

atoms are small compared to their inverse size, the interactions are effectively local and we can describe them with a local quantum field theory. Since we assume that all the atoms have the same spin we can describe them by a complex spin zero field:

$$\psi = \frac{1}{\sqrt{2}} [\psi_1 + i\psi_2]. \quad (1)$$

The symmetries are Galilean invariance and a global  $O(2)$ -symmetry. The Euclidean Lagrangian reads

$$\mathcal{L}_E = \psi^\dagger \partial_\tau \psi + \frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi - \mu \psi^\dagger \psi + g(\psi^\dagger \psi)^2 + \dots \quad (2)$$

Here,  $\mu$  is the chemical potential. The ellipses indicate all operators that are higher order in the number of fields  $\psi$  and derivatives and satisfy the  $O(2)$ -symmetry. The interaction  $g(\psi^\dagger \psi)^2$  represents  $2 \rightarrow 2$  scattering and the coupling constant  $g$  is proportional to the  $S$ -wave scattering length  $a$ :

$$g = \frac{2\pi a}{m}. \quad (3)$$

In the following we consider the dilute gas  $\rho a^3 \ll 1$ , which implies that we only need to retain the quartic interaction in the bare Lagrangian Eq. (2) [7]. We also set  $2m = 1$ .

In a field theoretic language, BEC is described as spontaneous symmetry breaking of the  $O(2)$ -symmetry and the complex field  $\psi$  acquires a nonzero vacuum expectation value  $v$ . The propagator is a  $2 \times 2$  matrix

$$\Delta(\omega_n, p) = \frac{1}{\omega_p^2 + \omega_n^2} \begin{pmatrix} \epsilon_p + V' & -\omega_n \\ \omega_n & \epsilon_p + V' + V''v^2 \end{pmatrix}, \quad (4)$$

where

$$\begin{aligned} V &= -\frac{1}{2}\mu v^2 + \frac{g}{4}v^4 \\ \epsilon_p &= p^2 \\ \omega_n &= 2\pi nT \\ \omega_p &= \sqrt{[\epsilon_p + V'(v) + V''(v)v^2][\epsilon_p + V'(v)]}, \end{aligned} \quad (5)$$

and primes denote differentiation with respect to  $v^2/2$ .

The grand canonical partition function has a path integral representation

$$Z = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-\int_0^\beta d\tau \int d^d x \mathcal{L}_E}. \quad (6)$$

In order to implement the renormalization group we modify the action by adding a piece containing a cutoff function,  $R_k^\Lambda(p)$ , to the action

$$S_{\beta,k}[\psi, \psi^\dagger] = \int_0^\beta d\tau \int d^d x \left\{ R_k^\Lambda \nabla \psi^\dagger \cdot \nabla \psi + S[\psi, \psi^\dagger] \right\}, \quad (7)$$

where

$$S[\psi, \psi^\dagger] = \int_0^\beta d\tau \int d^d x \mathcal{L}_E. \quad (8)$$

The function  $R_k^\Lambda(p)$  suppresses all modes in the path integral with momenta  $p$  less than  $k$  and larger than  $\Lambda$ , where  $k$  and  $\Lambda$  are our infrared and ultraviolet cutoffs, respectively. In the present work we choose a sharp cutoff such that

$$R_k^\Lambda(p) = \begin{cases} 0, & k < p < \Lambda, \\ \infty, & \text{otherwise.} \end{cases} \quad (9)$$

The flow equation or renormalization group equation for the effective action  $\Gamma_{\beta,k}[v]$  is obtained by taking a derivative with respect to the infrared cutoff  $k$  [11–13]. This equation specifies how  $\Gamma_{\beta,k}[v]$  changes as  $k$  is lowered:

$$\begin{aligned} \frac{\partial \Gamma_{\beta,k}}{\partial k} &= \frac{1}{2} T \sum_n \int \frac{d^d p}{(2\pi)^d} \left( \frac{\partial R_k^\Lambda}{\partial k} \right) \\ &\times \text{Tr} \left[ R_k^\Lambda \delta_{ab} + \frac{\partial^2 \Gamma_{\beta,k}}{\partial v_a \partial v_b} \right]^{-1}. \end{aligned} \quad (10)$$

The sum is over the Matsubara frequencies which take on the values  $2\pi nT$  for bosons and the integration is over  $d$ -dimensional momentum space. The trace is over internal indices. Also note that the flow equation explicitly depends on the cutoff function  $R_k^\Lambda$  (see also [14]).

The effective action can be expanded in powers of derivatives:

$$\begin{aligned} \Gamma_{\beta,k}[v] &= \int_0^\beta d\tau \int d^d x \left\{ U_{\beta,k}[v] + Z_{\beta,k}^{(1)}[v] \epsilon_{ij} v_i \partial_\tau v_j \right. \\ &\quad \left. + Z_{\beta,k}^{(2)}[v] (\nabla v_i)^2 + \dots \right\} \end{aligned} \quad (11)$$

To leading order in the derivative expansion the effective action  $\Gamma_{\beta,k}$  is (up to a volume factor) simply the effective

potential  $U_{\beta,k}$ , and  $Z_{\beta,k}^{(1)}$  and  $Z_{\beta,k}^{(2)}$  are both equal to unity. The RG-equation for the effective potential reads

$$k \frac{\partial U_{\beta,k}}{\partial k} = -\frac{S_d k^d}{2} \omega_k - S_d k^d T \ln [1 - e^{-\beta \omega_k}], \quad (12)$$

where

$$\omega_k(v) = \sqrt{[\epsilon_k + U'_{\beta,k} + U''_{\beta,k} v^2] [\epsilon_k + U'_{\beta,k}]} \quad (13)$$

$$S_d = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)}. \quad (14)$$

This equation will be derived in [15]. The first term on the right hand side of Eq. (12) is the  $T = 0$  contribution to the RG-equation and includes the quantum fluctuations. This equation interpolates between the bare theory for  $k = \Lambda$  and the physical theory at  $T \neq 0$  for  $k = 0$ , since we integrate out both quantum and thermal modes as we lower the cutoff. This implies that the boundary condition for the RG-equation is the *bare* potential. In Ref. [11] renormalization group ideas have been applied to relativistic  $\lambda\phi^4$ -theory using the real time formalism. Here, one can separate the propagator into a quantum and a thermal part, and the infrared cutoff is imposed only on the thermal part of the propagator. This implies that the theory interpolates between the physical theory at  $T = 0$  and the physical theory at  $T \neq 0$ . Hence, the boundary condition of the RG-equation is the physical effective potential at  $T = 0$  [11].

We have solved Eq. (12) for  $d = 3$  numerically for different values of the temperature and the result is displayed in Fig. 1. The curves clearly show a second order phase transition, which is expected from universality arguments.

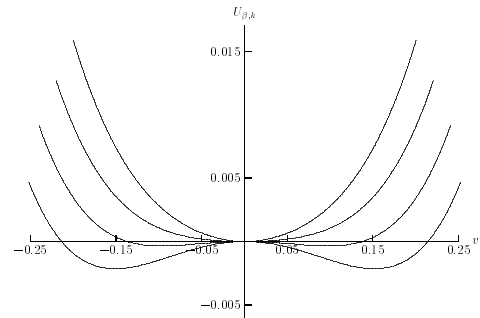


FIG. 1. The RG-improved effective potential  $U_{\beta,k}[v]$  for different values of the temperature. The phase transition is clearly second order.

The effective chemical potential  $\mu_{\beta,k}$  as well as the coupling constant  $g_{\beta,k}^{(4)}$  (defined as the discrete first and second derivatives of the effective potential with respect to  $v^2/2$ ) are displayed in Fig. 2. We see that both quantities vanish at the critical point. Moreover,  $g_{\beta,k}^{(6)}$  goes to

a non-zero constant at  $T_c$ . The inclusion of wavefunction renormalization effects turns the marginal operator  $g_{\beta,k}^{(6)}$  into an irrelevant operator that diverges at the critical temperature [12].

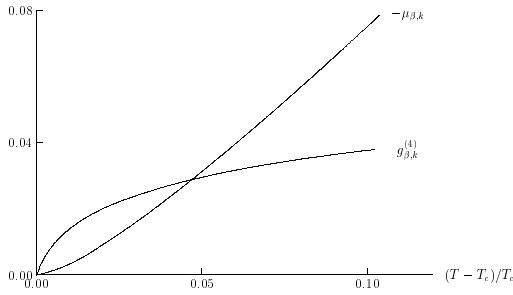


FIG. 2. The effective chemical potential  $\mu_{\beta,k}$  and the effective quartic coupling  $g_{\beta,k}^{(4)}$  near the critical temperature. Both vanish at  $T_c$ .

In order to locate the fixed points we write the effective potential in dimensionless form and make a series expansion around the origin:

$$\begin{aligned} \bar{U}_{\bar{\beta},k} &= \beta k^{-d} U_{\beta,k} \\ \bar{v} &= \beta^{1/2} k^{(2-d)/2} v \\ \bar{\omega}_k &= k^{-2} \omega_k \\ \bar{\beta} &= \beta k^2 \\ \bar{\mu}_{\bar{\beta},k} &= k^{-2} \mu_{\beta,k} \\ \bar{g}_{\bar{\beta},k}^{(4)} &= \beta^{-1} k^{d-4} g_{\beta,k}^{(4)} \\ &\vdots \\ \bar{U}_{\bar{\beta},k}(\bar{v}) &= \sum_{n=1}^{\infty} \frac{\bar{g}_{\bar{\beta},k}^{(2n)}}{n!} \left( \frac{\bar{v}^2}{2} \right)^n, \end{aligned} \quad (15)$$

and  $\bar{\mu}_{\bar{\beta},k} = -\bar{g}_{\bar{\beta},k}^{(2)}$ . Truncating the series after two terms, we obtain the following set of equations

$$k \frac{\partial \bar{\mu}_{\bar{\beta},k}}{\partial k} = -2\bar{\mu}_{\bar{\beta}} + S_d \bar{\beta} \bar{g}_{\bar{\beta},k}^{(4)} \coth[\bar{\beta}(1 - \bar{\mu}_{\bar{\beta},k})/2] \quad (16)$$

$$\begin{aligned} k \frac{\partial \bar{g}_{\bar{\beta},k}^{(4)}}{\partial k} &= (d-4)\bar{g}_{\bar{\beta},k}^{(4)} + S_d \bar{\beta} [\bar{g}_{\bar{\beta},k}^{(4)}]^2 \\ &\times \left[ \frac{1}{2(1 - \bar{\mu}_{\bar{\beta},k})} \coth[\bar{\beta}(1 - \bar{\mu}_{\bar{\beta},k})/2] \right. \\ &\left. + \frac{4\bar{\beta} e^{\bar{\beta}(1 - \bar{\mu}_{\bar{\beta},k})}}{(e^{\bar{\beta}(1 - \bar{\mu}_{\bar{\beta},k})} - 1)^2} \right]. \end{aligned} \quad (17)$$

A similar set of equations has also been obtained in Ref. [7] by considering the one-loop diagrams that contribute to the running of the different vertices (They use

the operator formalism and normal ordering so the zero temperature part of the tadpole vanishes).

Expanding in powers of  $\bar{\beta}(1 - \bar{\mu}_{\bar{\beta},k})$  and introducing the variables

$$r = \frac{\bar{\mu}_{\bar{\beta},k}}{1 - \bar{\mu}_{\bar{\beta},k}}, \quad s = \frac{\bar{g}_{\bar{\beta},k}^{(4)}}{(1 - \bar{\mu}_{\bar{\beta},k})^2}, \quad (18)$$

the RG-equations can be written as

$$\frac{\partial r}{\partial k} = -2[1 + r][r - S_d s] \quad (19)$$

$$\frac{\partial s}{\partial k} = -s[\epsilon + 4r - 9S_d s]. \quad (20)$$

Here,  $\epsilon = 4 - d$ . We have the trivial Gaussian fixed point  $(0, 0)$  as well as the infinite temperature Gaussian fixed point  $(-1, 0)$ . Finally, for  $\epsilon > 0$  there is the infrared Wilson-Fisher fixed point  $(\epsilon/5, \epsilon/(5S_d))$  [9].

One can now calculate the critical exponent  $\nu$  which is related to the correlation length  $\xi$  through

$$\xi \sim |T - T_c|^{-\nu}. \quad (21)$$

Linearizing around the fixed point, we find the eigenvalues  $(-1.278, 1.878)$ . This implies that the critical exponent is  $\nu = 0.532$  in agreement with the result of Biljsma and Stoof [7].

In order to check the convergence of this expansion, we have repeated the calculation including more terms in the Taylor expansion of the effective potential. The critical exponent  $\nu$  as a function of the number of terms  $N$  is shown in Fig. 3. The critical exponent  $\nu$  fluctuates around the value 0.73 and should be compared to experiment ( $^4\text{He}$ ) and the  $\epsilon$ -expansion which both give a value of 0.67 [16]. Moreover, our result agrees with that of Morris, who considered the relativistic  $O(2)$ -model in  $3d$  at zero temperature [13]. The fact that the critical exponent oscillates as a function of  $N$  reflects that the polynomial expansion of the effective potential breaks down near the critical temperature [12,13,17].

To improve our results for the critical exponents, we must go beyond leading order in the derivative expansion.

We obtain the same critical behaviour as in the three-dimensional  $O(2)$ -theory as expected from universality. This can be seen directly, without computation, by considering the dimensionless form of Eq. (12):

$$\begin{aligned} 0 &= \left[ k \frac{\partial}{\partial k} - \frac{1}{2}(d-2)\bar{v}\partial_{\bar{v}} + d \right] \bar{U}_{\bar{\beta},k} \\ &+ \frac{S_d}{2} \bar{\beta} \bar{\omega}_k + S_d \ln \left[ 1 - e^{-\bar{\beta} \bar{\omega}_k} \right]. \end{aligned} \quad (22)$$

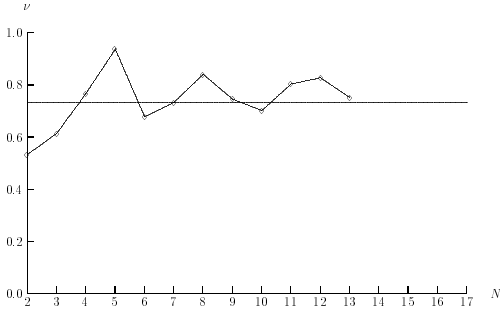


FIG. 3. The critical exponent  $\nu$  as a function of number of terms  $N$  in the polynomial expansion.

The critical potential is now found by demanding that [13]

$$k \frac{\partial \bar{U}_{\bar{\beta},k}}{\partial k} = 0, \quad (23)$$

and then expanding in powers of  $\bar{\beta}\bar{\omega}_k$ :

$$\left[ -\frac{1}{2}(d-2)\bar{v}\partial_{\bar{v}} + d \right] \bar{U}_{\bar{\beta},k} = -\frac{S_d}{2} \bar{\beta}\bar{\omega}_k - S_d \ln [\bar{\beta}\bar{\omega}_k]. \quad (24)$$

Taking the limit  $\bar{\beta} \rightarrow 0$  [12] and ignoring the piece which is independent of  $v$ , this leads to

$$\left[ -\frac{1}{2}(d-2)\bar{v}\partial_{\bar{v}} + d \right] \bar{U}_{\bar{\beta},k} = -\frac{S_d}{2} \left[ \ln [1 + \bar{U}'] + \ln [1 + \bar{U}' + \bar{U}''\bar{v}^2] \right]. \quad (25)$$

This is exactly the same equation as obtained by Morris for an  $O(2)$ -symmetric scalar theory in three dimensions to leading order in the derivative expansion [13]. Therefore, the results for the critical behaviour at leading order in the derivative expansion will be the same as those obtained in the three-dimensional  $O(2)$ -model at zero temperature.

In the present letter we have explicitly demonstrated that the phase transition of the homogeneous non-relativistic Bose gas is second order and that the critical behaviour is the same as in the three-dimensional scalar  $O(2)$ -model. Both properties are expected from universality arguments.

One natural extension of the present work is to include wave function renormalization effects by going to second order in the derivative expansion and investigate noncritical quantities such as the critical temperature and the superfluid fraction in the system.

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